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Jump-start in the analytic perturbation procedure for the Friedrichs model

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Abstract

For the Friedrichs model, obtained as a one-dimensional ε-perturbation $\mathcal{P}_\varepsilon$ of the orthogonal sum of the momentum $\mathcal{P} = \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}$ and a finite Hermitian matrix: $P \oplus A \longrightarrow \mathcal{P}_\varepsilon$, the scattering matrix is presented as $S_\varepsilon(p) = [I + \mathrm{i}\varepsilon M(p)]^{-1}[I - \mathrm{i}\varepsilon M(p)]$. The rational Nevanlinna-class Krein–Weyl function M is associated with the operator A and has poles at the eigenvalues of A. It is proven that for any eigenvalue α_0 of A there exists an *intermediate operator* $\mathcal{P}_0^\varepsilon$, which is constructed as a one-dimensional perturbation $\mathcal{P}_\varepsilon^0$ of the orthogonal sum $\mathcal{P} \oplus A_0^\varepsilon$ of the momentum and an appropriate one-dimensional operator—a solvable model, which plays the role of an intermediate operator in the scattering problem to the pair $(\mathcal{P}_\varepsilon, \mathcal{P})$. The scattering matrix S_ε^0 to the pair $\left(\mathcal{P}_\varepsilon, \mathcal{P}_\varepsilon^0\right)$ is an analytic function of ε and the total scattering matrix to the pair $(\mathcal{P}_\varepsilon, \mathcal{P})$ can be factorized as a product

$$S_\varepsilon(p) = S_\varepsilon^0 S_0^\varepsilon,$$

where S_0^ε is the scattering matrix to the pair $\mathcal{P}_0^\varepsilon, \mathcal{P}$. It is represented by a single Blaschke factor with a pole and zero approaching α_0 when $\varepsilon \to 0$. The non-analytic factor $\mathcal{S}_0^\varepsilon$ describes creation of the resonance from the eigenvalue α_0 of the operator A.

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1. Introduction

The standard technique of the analytic perturbation theory is developed for additive perturbations $\mathcal{A}_\varepsilon = \mathcal{A} + \varepsilon\mathcal{B}$ of operators with discrete spectrum, see, for instance, [1]. It is well known, see [2], that for operators with continuous spectrum the analytic perturbation procedure is convergent only for small values of the perturbation parameter ε—'below the

threshold of creation of resonances', but cannot be extended beyond this limit. Poincaré connected the divergence of the analytic perturbation procedure in celestial mechanics with resonances, which appear in [3] as small denominators. Following Poincaré, Prigogine discussed the divergence of the perturbation procedure on a continuous spectrum in connection with irreversibility of dynamics, and he even tried to attract the attention of specialists to the importance of extending the range of application of the analytic perturbation technique beyond the natural limit. He also attempted to find an *intermediate operator* and use it as a first step—a sort of a 'jump-start'—in the analytic perturbation procedure for operators with continuous spectrum. He assumed that there may exist an operator $\mathcal{C}_\varepsilon$ in the commutant of the non-perturbed operator, see [4–8], such that the analytic perturbation procedure is convergent for the pair $(\mathcal{C}_\varepsilon, \mathcal{A}_\varepsilon)$. No such intermediate operator was found, and the idea of the jump-start was abandoned. It appeared again in scattering problems on *Quantum Networks*, see [9]. The simplest quantum network, the *Quantum Switch*, see [10, 11], is constructed as a circular quantum well Ω_0 with four straight wires Ω_s width δ, attached to it orthogonally. The intermediate operator for the scattering problem on the quantum network can be constructed via Glazman's 'splitting' by imposing a semi-transparent boundary condition on some sections of the semi-infinite wires. This boundary condition splits the original Hamiltonian into two parts: the trivial part in open channels and the non-trivial part on the joining of closed channels and the compact part of the network. This split operator plays the role of the intermediate operator for the quantum network. In contrast to the assumption of Prigogine, this operator does not commute with the non-perturbed operator, and, more importantly, it is defined locally—only for the selected spectral band, where the multiplicity of the continuous spectrum is constant.

For operators with Lebesgue's absolute continuous spectrum (constant multiplicity) the above construction of the intermediate is not applicable. But the question of convergence of the analytic perturbation procedure is actual: the standard procedure of analytic perturbation theory suggested in [2] for the additive perturbation of the momentum operator ('Friedrichs model', see for instance [13–19])

$$\mathcal{P}_\varepsilon = \mathcal{P} + \varepsilon A$$

is convergent only for small values of the perturbation parameter ε. Extension of the perturbation procedure beyond the threshold of creation of resonances requires the non-analytic techniques of the *mathematical scattering theory*, see, for instance, [20–22].

In this paper, we describe an observation which probably will help to extend the technique of jump-start to general operators with Lebesgue spectrum, via constructing for them a modified analytic perturbation procedure in two steps:

$$\mathcal{P} \longrightarrow \mathcal{P}^0{}_\varepsilon \longrightarrow \mathcal{P}_\varepsilon.$$

In the first step, the scattering matrix consists of a single Blaschke factor which is *non-analytic* with respect to ε. The scattering matrix in the second step is analytic with respect to ε.

Our paper has the following plan. In section 2 we describe our version of the Friedrichs model with non-trivial 'inner structure' and calculate the corresponding scattering matrix in terms of the corresponding Krein–Weyl function defined by the spectral characteristics of the inner Hamiltonian A. In section 3 we explore the distribution of resonances and factorize the corresponding scattering matrix as a product of the *non-analytic factor* S_0^ε and an *analytic factor* S_ε^0 and interpret S_0^ε as the scattering matrix of the intermediate operator $\mathcal{P}_0^\varepsilon$ with respect to $\mathcal{P}$.

In the appendix, we quote the Gohberg–Sigal theorem on the logarithmic residue, which serves the foundation of the factorization of the scattering matrix.

It is interesting to see that resonances appear as an essential detail of our construction, in agreement with the anticipation of Poincaré [3] and Prigogine. But, unfortunately, the construction of the intermediate operator in the first step requires *exact data* of the resonance. In Friedrichs model, we may obtain these data via solving some algebraic equation. But in the general case, substitution of the exact data by the *approximate data* destroys the analyticity of the complementary scattering matrix obtained in the second step.

2. Friedrichs model with an inner structure

We use in this paper the Friedrichs model with an inner structure, which is obtained as a perturbation $\mathcal{P}_\beta$, see below (2), of the orthogonal sum $\mathcal{P} \oplus A$ of the momentum $\mathcal{P} = \frac{1}{\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}x}$ in $L_2(R, E)$, $\dim E = m < \infty$ and a finite-dimensional self-adjoint operator acting in the space K, $A : K \to K$, $\dim K = k < \infty$.

We construct $\mathcal{P}_\beta$ via the symplectic operator extension procedure, see [23], beginning from restriction of the momentum $\mathcal{P} \to \mathcal{P}_0$ onto the domain $D_0 = W_2^{1,0}(R, E)$ of all smooth functions taking values in E and vanishing at the origin. The operator $\mathcal{P}_0$ is symmetric, with deficiency indices (m, m), $m = \dim E$. The corresponding adjoint operator $\mathcal{P}_0^+$ is defined on $W_2^1(R_-) \oplus W_2^1(R_+)$, without any boundary conditions at the origin. The boundary form, see for instance [24–26] of the adjoint operator $J(u, v) = \langle \mathcal{P}_0^+ u, v\rangle - \langle u, \mathcal{P}_0^+ v\rangle = \mathrm{i}[u\bar{v}(0^+) - u\bar{v}(0^-)]$ can be represented in terms of the corresponding symplectic variables $\xi_\pm$,

$$\xi_+^u = \frac{u(0^+) + u(0^-)}{2}, \qquad \xi_-^u = \mathrm{i}[u(0^+) - u(0^-)]$$

as

$$J_p(u, v) = \langle \xi_-^u, \xi_+^v\rangle_E - \langle \xi_+^u, \bar{\xi}_-^v\rangle_E. \tag{1}$$

A version of operator extension theory for non-densely defined operators was developed in [27]. The symplectic form of it is described in [26, 28]. The main obstacle to the extension construction procedure in the finite-dimensional case—the absence of the adjoint operator—is avoided by reducing the construction of the extension onto the defect—the sum of deficiency subspaces $N_i = N$, $N_{-i} = \frac{A+\mathrm{i}I}{A-\mathrm{i}I}N$, $\dim N = n \leqslant k/2$, if $N_i \cap N_{-i} = 0$, which is automatically fulfilled if $n = 1$. We derive an expression for the scattering matrix in the case when $\dim N = n \leqslant k/2$. In fact, the derived expression for the scattering matrix is valid also in the case $\dim K = k = 1$, see [31, 32], and it can be verified by a direct calculation in our case too. We use the explicit formula for the scattering matrix in the case $k = 1$ in section 3, theorem 3.4.

The restricted operator is defined on the non-dense domain $D_{A_0} = \frac{1}{A-\mathrm{i}I}K \ominus N$. Choosing an orthogonal basis $\mathbf{e}_s \in N = N_i$ and the corresponding basis $\hat{\mathbf{e}}_s = \frac{A+\mathrm{i}I}{A-\mathrm{i}I}\mathbf{e}_s \in N_{-i}$, we introduce a new basis in defect $D = N + N_{-i}$:

$$W_s^+ = \frac{\mathbf{e}_s + \hat{\mathbf{e}}_s}{2} = \frac{A}{A - \mathrm{i}I}\mathbf{e}_s, \qquad W_s^- = \frac{\mathbf{e}_s - \hat{\mathbf{e}}_s}{2\mathrm{i}} = -\frac{1}{A - \mathrm{i}I}\mathbf{e}_s.$$

Then the elements from the defect are uniquely presented as linear combinations: $u_d = \frac{A}{A-\mathrm{i}pI}\eta_+ - \frac{1}{A-\mathrm{i}pI}\eta_-$, where

$$\eta_+ = \sum_s \eta_+^s \mathbf{e}_s, \qquad \eta_- = \sum_s \eta_-^s \mathbf{e}_s, \qquad \eta_\pm \in N.$$

The formal adjoint operator A^+ is defined on the defect as

$$A^+\mathbf{e}_s = -\mathrm{i}\mathbf{e}_s, \qquad A^+\hat{\mathbf{e}}_s = \mathrm{i}\hat{\mathbf{e}}_s$$

or

$$A^+W_s^+ = W_s^-, \qquad A^+W_s^- = -W_s^+.$$

Lemma 2.1 [24, 28]**.** *The boundary form of the formal adjoint operator is calculated in terms of symplectic variables $\eta_\pm^u$, $\eta_\pm^v$ as*

$$J_A(u, v) = \langle A^+u, v\rangle - \langle u, A^+v\rangle = \left\langle \eta_+^u, \eta_-^v \right\rangle - \left\langle \eta_-^u, \eta_+^v \right\rangle$$

and it depends only on the parts of the vectors u, v in the defect.

Consider the orthogonal sum $P_0 \oplus A_0$ of the restricted operators, and construct a Lagrangian plane $\mathcal{L}_\beta$ parametrized by the Hermitian matrix B connecting the symplectic coordinates $\xi_\pm$ of the 'outer' component with the symplectic coordinates $\eta_\pm$ of the 'inner' component

$$B = \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix}$$

with elements $\beta_{10}^+ = \beta_{01} \in C_m \times C_n$, $\beta_{00} \in C_m \times C_m$, $\beta_{11} \in C_n \times C_n$. In the classical case when $\dim E = 1$ the scattering matrix is a scalar function. The stationary scattering matrix is unitary on the real axis. Though the final result, theorem 3.4, for the one-dimensional deficiency subspaces, is most interesting, we continue our analysis in general case, referring to the one-dimensional case if necessary.

Theorem 2.2. *The joint boundary form $J_p(u, v) + J_A(\mathbf{u}, \mathbf{v})$ vanishes on the Lagrangian plane $\mathcal{L}_\beta$ described by the equation*

$$\begin{pmatrix} \xi_- \\ \eta_+ \end{pmatrix} = B \begin{pmatrix} \xi_+ \\ \eta_- \end{pmatrix}. \tag{2}$$

This Lagrangian plane defines the corresponding self-adjoint extension P_β of $P_0 \oplus A_0$.

The constructed operator $\mathcal{P}_\beta$ has absolutely continuous spectrum multiplicity m on the interval $(-\infty, \infty)$. The corresponding eigenfunctions $\mathbf{\Psi}$ have two components: in the 'outer' space $L_2(R, E)$ and in the inner space K, $\mathbf{\Psi} = \{\Psi_0, \Psi_1\}$. They fulfil the adjoint homogeneous equations and the above boundary conditions (2). The symplectic coordinates $\eta_\pm$ of the solution are connected via the corresponding Krein function: $(n \times n)$ matrix function $\mathcal{M}(p) = P\frac{I+pA}{A-pI}P$, where P is the orthogonal projection $P = P^+ : K \to N$, see [29, 23]:

$$\eta_- = -\mathcal{M}(p)\eta_+.$$

The Krein function of the inner operator A_0 is an abstract analogue of the Weyl–Titchmarsh function, which is an important tool in spectral theory of differential operators, see [30]. Further, we call it Krein–Weyl function. It belongs to Nevanlinna class (i.e., it is analytic and has a positive imaginary part in the upper half-plane $\operatorname{Im} p > 0$). Then, we present the outer component of the 'incoming' eigenfunction by the ansatz

$$\Psi_0(x, \nu) = \begin{cases} \mathrm{e}^{\mathrm{i}px}\nu & \text{for} \quad x < 0, \\ \mathrm{e}^{\mathrm{i}px}S\nu & \text{for} \quad x > 0, \end{cases} \tag{3}$$

and the inner component defined as

$$\Psi_1(\nu) = \frac{A + \mathrm{i}I}{A - pI}\eta_+(\nu), \tag{4}$$

with $S, \eta_+(\nu)$ to be found from the above boundary condition (2). We can rewrite the equation (2) as

$$\begin{pmatrix} \mathrm{i}(S\nu - \nu) \\ \eta_+(\nu) \end{pmatrix} = \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & \beta_{11} \end{pmatrix} \begin{pmatrix} \frac{S+1}{2}\nu \\ -\mathcal{M}(p)\eta_+(\nu) \end{pmatrix},$$

hence $\eta_+(\nu) = \beta_{10}\frac{I+S}{2}\nu - \beta_{11}\mathcal{M}\eta_+(\nu)$ and

$$\mathrm{i}(S-1)\nu = [\beta_{00} - \beta_{01}\mathcal{M}(1+\beta_{11}\mathcal{M})^{-1}\beta_{01}]\frac{S+I}{2}\nu$$

and

$$\eta_\nu = \frac{1}{I+\beta_{11}\mathcal{M}}\beta_{10}\frac{I+S}{2}\nu.$$

This implies the following expression for the scattering matrix $S = S_\beta(p)$:

Theorem 2.3. *The scattering matrix defined as the transmission coefficient S in the exterior component of the scattered waves (3) is represented as*

$$S_\beta(p) = \frac{\mathrm{i}I + \frac{1}{2}[\beta_{00} - \beta_{01}\mathcal{M}(I+\beta_{11}\mathcal{M})^{-1}\beta_{10}]}{\mathrm{i}I - \frac{1}{2}[\beta_{00} - \beta_{01}\mathcal{M}(I+\beta_{11}\mathcal{M})^{-1}\beta_{10}]}. \tag{5}$$

The scattered waves of the perturbed operator are defined by (3) with $S = S_\beta(p)$.

Remark. The perturbed momentum operator P_β acting in extended space $L_2(R, E) \oplus K$, is unitary equivalent to P. This unitary equivalence is defined by the corresponding wave operators, transforming the non-perturbed scattered waves into the perturbed ones

$$W_-\,\mathrm{e}^{\mathrm{i}px}\nu = \begin{pmatrix}\Psi_0\\ \Psi_1\end{pmatrix}, \qquad \mathcal{P}_\beta = W_-\mathcal{P}W_-^+.$$

The same operators can transform the multiplication operator $Q : u \longrightarrow xu$ into

$$\mathcal{Q}_\beta = W_-\mathcal{Q}W_-^+,$$

acting in the extended space, such that the pair $\mathcal{P}_\beta$, $\mathcal{Q}_\beta$ fulfils the same commutation relations as $\mathcal{P}$, $\mathcal{Q}$. This fact permits us to introduce the corresponding creation and annihilation operators, coherent states and other standard objects. We postpone discussion of this matter to the forthcoming paper.

Remark 2. If $B = 0$, then $S = 1$, which corresponds to the non-perturbed operator. But *it is impossible* to construct an analytic (with respect to the perturbation parameters β_{il}) branch of eigenfunctions $\Psi_\nu(p, \beta)$ of the perturbed operator for any p that coincides with the eigenfunction $\mathrm{e}^{\mathrm{i}px}\nu$ of the non-perturbed operator at $B = 0$. In the following section, we will suggest a special perturbation procedure which allows us to overcome this basic difficulty locally, near a certain point $(p_0, 0)$ in the space (p, β) based on the introduction of a special *intermediate operator*.

Generally, the above formula (5) produces an expression for the scattering matrix with generally non-trivial asymptotic behaviour when $p \to \infty$:

$$S_\beta(p) \to \frac{\mathrm{i}I + \frac{1}{2}[\beta_{00} - \beta_{01}(-PAP)[I+\beta_{11}(-PAP)]^{-1}\beta_{10}]}{\mathrm{i}I - \frac{1}{2}[\beta_{00} - \beta_{01}(-PAP)[I+\beta_{11}(-PAP)]^{-1}\beta_{10}]} \neq I.$$

Theorem 2.4. *The scattering matrix has the 'natural' asymptotic behaviour at infinity, $S_\beta(p) \to I$, if and only if $[\beta_{00} - \beta_{01}(-PAP)(1+\beta_{11}(-PAP))^{-1}\beta_{10}] = 0$. In particular, the scattering matrix tends to I at infinity if*

$$\beta_{11} = 0 \quad \textit{and} \quad \beta_{00} + \beta_{01}PAP\beta_{10} = 0. \tag{6}$$

If the boundary parameters are selected such that the condition (6) is fulfilled, then the corresponding scattering matrix is represented as a finite Blashke–Potapov product, see [33, 34], with zeros (resonances) in the upper half-plane $\operatorname{Im} p > 0$.

The proof is based on the decomposition

$$\mathcal{M}(p) = -PAP + P\frac{I + A^2}{A - pI}P := a + m(p),$$

with the Nevanlinna-class matrix function m tending to 0 at infinity. In particular, the scattering matrix tends to I at infinity if $\beta_{11} = 0$ and $\beta_{00} + \beta_{01}PAP\beta_{10} = 0$. In this case, the expressions in brackets in both numerator and denominator are Nevanlinna functions which tend to 0 at infinity and the scattering matrix is unitary on the real axis. To prove the Blashke–Potapov decomposition, we need to check that the projection which corresponds to left regular factor at the zero p_s coincides with the residue of the singular factor at the pole $\bar{p}_s$. It is true, according to symmetry principle, because the scattering matrix is unitary on the real axis. Then the zero p_s and the pole $\bar{p}_s$ can be collected into one factor, which gives the representation of the rational expression

$$S_\beta(p) = \frac{2\mathrm{i} - \beta_{01}m\beta_{10}}{2\mathrm{i} + \beta_{01}m\beta_{10}} = \frac{2\mathrm{i} - \beta_{01}\frac{I+A^2}{A-pI}\beta_{10}}{2\mathrm{i} - \beta_{01}\frac{I+A^2}{A-pI}\beta_{10}} \tag{7}$$

with vector zeros (p_s, ν_s) with *root vectors* $\nu_s S_\beta(p_s)\nu_s = 0$ in the upper half-plane $\operatorname{Im} p_s > 0$, in form of the Blaschke–Potapov product:

$$S_\beta(p) = \prod_s \left[\frac{p - p_s}{p - \bar{p}_s}P_s + (I - P_s)\right].$$

The orthogonal projections P_s in E depend on the order of factors, see the discussion in the end of next section.

Note that the constructed model has all features of the Lax–Phillips scattering system, see [35], and it can serve as a simplest nontrivial model of such a system.

3. Analytic perturbation procedure and the intermediate operator

We begin with a general statement concerning resonances.

Lemma 3.1. *If the condition (6) is fulfilled, then the zeros of the scattering matrix resonances depend analytically on the boundary parameter β_{01} and may be found for small values of the parameter via analytic perturbation procedure.*

Proof. It is based on the matrix version of Rouché theorem by Gohberg and Sigal, see [36] and the appendix below, where the simplest version of this general fact is quoted. We consider here only the generic case when all eigenvalues α_s of the operator A are simple. Denoting by $q_s = e_s\rangle\langle e_s$ the corresponding eigenprojections, we may present the function in the numerator of the scattering matrix as

$$\beta_{01}P\frac{I + A^2}{A - pI}P\beta_{01} = \sum_s \frac{1 + \alpha_s^2}{\alpha_s - p}\varepsilon_s Q_s, \tag{8}$$

where $Q_s = \nu_s\rangle\langle\nu_s$ is an orthogonal projection onto the one-dimensional subspace spanned by $\beta_{01}Pe_s = \|\beta_{01}Pe_s\|\nu_s$ and $\varepsilon_s = \|\beta_{01}Pe_s\|^2$. We assume that $\varepsilon = (\varepsilon_0, \varepsilon_1, \varepsilon_3, \ldots)$ is a non-zero vector. We will use ε as a perturbation parameter instead of the matrix β_{01}. Our immediate aim is to calculate the resonance p_{s0} created from α_0 at $\varepsilon_0 = 0$, assuming that $|\varepsilon| = \max \varepsilon_s$ is small.

It is clear that an essential contribution to the above function (8) near the pole α_0 is defined by the nearest singular summand $\frac{1+\alpha_0^2}{\alpha_0-p}\varepsilon_0 Q_0$. Planning to use the Gohberg–Sigal theorem, see the appendix, we introduce two functions

$$m(p) = 2\mathrm{i} - \sum_s \frac{1+\alpha_s^2}{\alpha_s - p}\varepsilon_s Q_s \quad \text{and} \quad m_0(p) = 2\mathrm{i} - \frac{1+\alpha_0^2}{\alpha_0 - p}\varepsilon_0 Q_0,$$

$$m_0^{-1}(p) = \frac{1}{2\mathrm{i}}\left(I - Q_0 \frac{\mathrm{i}\varepsilon_0 \frac{1+\alpha_0^2}{2}}{\alpha_0 + \mathrm{i}\varepsilon_0 \frac{1+\alpha_0^2}{2} - p}\right),$$

and their ratio

$$m_0^{-1}(p)m(p) = I - m_0^{-1}\sum_{s\neq 0} \frac{1+\alpha_s^2}{\alpha_s - p}\varepsilon_s Q_s := I - m_0^{-1}\mathbf{m}_0.$$

Zeros of the function m coincide with resonances. The only zero of the function m_0 sits at $\alpha_0(\varepsilon) = \alpha_0 + \mathrm{i}\varepsilon_0\frac{1+\alpha_0^2}{2}$. Consider a circle Σ_0 with radius δ centred at $\alpha_0(\varepsilon)$. The ratio $m_0^{-1}\mathbf{m}_0$ can be estimated on the circle $\Sigma_0 = \left\{p : \left|\alpha_0 + \mathrm{i}\frac{\varepsilon_0}{2} - p\right| = \delta\right\}$ as

$$\left\|m_0^{-1}\mathbf{m}_0\right\| \leqslant \frac{1}{2}\left[1 + \varepsilon_0\frac{1+\alpha_0^2}{2}\right]\sum_{s\neq 0}\frac{\varepsilon_s}{|\alpha_s - \alpha_0| - \delta}, \tag{9}$$

hence it is small for small $|\varepsilon| + \delta \ll \min|\alpha_0 - \alpha_s|$. Both functions m, m_0 are analytic inside the circle Σ_0, hence, due to Gohberg–Sigal theorem, the function m has zeros inside Σ_0 with the total multiplicity $M_0 = \dim Q_0$. In particular, it has only one simple zero, if $M_0 = 1$.

We continue our reasoning by assuming that $M_0 = 1$. Then the function $m^{-1} = \left[I - m_0^{-1}\mathbf{m}_0\right]^{-1}m_0^{-1}$ has only one pole $p_0(\varepsilon)$, which will be found by integration of m^{-1} on the circle Σ_0.

Consider the left factorization of the function m at the resonance $p_0(\varepsilon)$:

$$m(p) = \left([p - p_0(\varepsilon)]\mathbf{P}_0^+(\varepsilon) + b(\varepsilon)\left(I - \mathbf{P}_0^+\right)\right)\hat{\mu}(p) := m_0^+\hat{\mu}(p). \tag{10}$$

Here $\mathbf{P}_0^+$ is the orthogonal projection onto the null-space of m^+ at the point $p_0(\varepsilon)$ ('left' null-space of m : $\mathbf{P}_0^+ m\left(p_0(\varepsilon)\right) = 0$. The residue of the function m^{-1} at $p_0(\varepsilon)$ is calculated as an integral of $m^{-1} = \left[I - m_0^{-1}\mathbf{m}_0\right]^{-1}m_0^{-1} = \hat{\mu}^{-1}m_0^{+-1}$ on the circle. The representation (10) gives the formula

$$\frac{1}{\hat{\mu}\left(p_0(\varepsilon)\right)}\mathbf{P}_0^+(\varepsilon) = \frac{1}{2\pi}\oint_{\Sigma_0}\frac{1}{\hat{\mu}(p)}\left(\frac{\mathbf{P}_0^+(\varepsilon)}{[p - p_0(\varepsilon)]} + \frac{\left(I - \mathbf{P}_0^+(\varepsilon)\right)}{b(\varepsilon)}\right)\mathrm{d}p. \tag{11}$$

On the other hand, the residue can also be found via integration of another expression for $m^{-1}(p)$ on the circle:

$$\frac{1}{2\pi}\oint_{\Sigma_0}\left[(m_0(p))^{-1} + (m_0(p))^{-1}\mathbf{m}_0(p)(m_0(p))^{-1} + \cdots\right]\mathrm{d}p. \tag{12}$$

The series in the integrand is converging geometrically, and each term of it is calculated by residues at the pole $\alpha_0\varepsilon$. For instance,

$$\begin{aligned}\frac{1}{2\pi}\oint_{\Sigma_0}(m_0(p))^{-1}\,\mathrm{d}p &= \frac{1}{2\pi}\oint_{\Sigma_0}\left\{\frac{1}{2\mathrm{i}}\left[\frac{\alpha_0 - p}{\alpha_0 + \mathrm{i}\frac{\varepsilon_0\left(1+\alpha_0^2\right)}{2} - p}\right]Q_0 + \frac{1}{2\mathrm{i}}\left[I - Q_0\right]\right\}\mathrm{d}p \\ &= \varepsilon_0\frac{1+\alpha_0^2}{2}Q_0.\end{aligned}$$

The next terms contain derivatives of $\mathbf{m}_0$ at the pole $\alpha_0\varepsilon = \alpha_0 + \mathrm{i}\frac{\varepsilon_0(1+\alpha_0^2)}{2}$. The structure of the whole expansion is similar to the expansion by residues which arises in the standard Feynman diagram technique, because the idea of calculation of the residue based on the use of two different forms of the integrand is the same as in the Feynman case, see [37]. Comparing the series (12 with 11) we obtain the representation of $\mathbf{P}_0^+$ in the form of geometrically convergent series.

The analyticity of the projection $\mathbf{P}_0^+(\varepsilon)$ as a function of ε follows from the geometrical convergence of the series in the integrand of (12).

The orthogonal projection $P_0^+(\varepsilon_0)$ onto the null-subspace of m^+ $(p_0(\varepsilon_0))$ is calculated, up to a constant, as

$$\mathbf{P}_0^+(\varepsilon)\left(\frac{1}{\hat{\mu}-(p_0(\varepsilon))}\right)^+\frac{1}{\hat{\mu}-(p_0(\varepsilon))}\mathbf{P}_0^+(\varepsilon) = \mathrm{Const}\,\mathbf{P}_0^+(\varepsilon).$$

The zero $p_0(\varepsilon)$ of the function m can be obtained from the comparison of the previous integral with the integral

$$\frac{1}{2\pi}\oint_{\Sigma_0}\frac{p}{\hat{\mu}-(p)}\,\mathrm{d}p = \frac{p_0(\varepsilon)}{\hat{\mu}-(p_0(\varepsilon))}\mathbf{P}_0^+(\varepsilon).$$

Thus both the resonance $p_0(\varepsilon)$ and the corresponding left root-vector ν_0^+ are defined as analytic functions of the boundary parameters. The right root-vector can be found in a similar way.

□

Assuming that the condition of the preceding theorem is fulfilled, consider the rational representation of the stationary scattering matrix of the operator $\mathcal{P}_\beta$

$$S_\beta(p) = \frac{2\mathrm{i} - \sum_s \frac{1+\alpha_s^2}{\alpha_s - p}\varepsilon_s Q_s}{2\mathrm{i} + \sum_s \frac{1+\alpha_s^2}{\alpha_s - p}\varepsilon_s Q_s} \tag{13}$$

where Q_s is the orthogonal projection onto the subspace spanned by the vector $\beta_{01}P_N\nu_s$ obtained from the eigenvector ν_s of A via projection onto the deficiency subspace $N = N_i$, followed by application of the linear boundary map β_{01}. The scattering matrix is presented, for a small ε, in the form of Blashke–Potapov product with simple Blaschke factors B_s,

$$\prod_s \left\{\left[\frac{p-p_s(\varepsilon)}{p-\bar{p}_s(\varepsilon)}\right]P_s + P_s^\perp\right\} =: \prod_s B_s. \tag{14}$$

Theorem 3.2. *Each Blaschke factor in (14) is an analytic function of both variables (p, ε) on the product of a small neighbourhood of the origin in ε-space and a complement of a small neighbourhood of the corresponding point α_s of creation of the resonance $p_s(\varepsilon)$ in p-space. In particular, selecting $B_0 = S_0^\varepsilon$ as the first factor on the right-hand side, we obtain the factorization of the scattering matrix in the form of two factors,*

$$S_\varepsilon = S_\varepsilon^0 S_0^\varepsilon, \tag{15}$$

with the left factor analytic with respect to (ε, p) on a small neighbourhood of $(0, \alpha_0)$, and the right factor non-analytic on that neighbourhood, since $p_0(\varepsilon)$, $\bar{p}_0(\varepsilon) \to \alpha_0$ if $\varepsilon \to 0$.

The non-analyticity of the factor S_0^ε causes the non-analyticity of the whole product, and it corresponds to the fact of the non-analyticity of the scattering matrix with respect to the perturbation parameter at the 'threshold of creation of resonances'.

Remark (construction of Blaschke factors)**.** If the scattering matrix is unitary on the real axis, then the right Blaschke factors are constructed at each resonance

$$S_\beta(p) = B^s(p)\mathbf{B}_s(p), \qquad \mathbf{B}_s(p) = \frac{p - p_s(\varepsilon)}{p - \bar{p}_s(\varepsilon)}\mathbf{P}_s + [I - \mathbf{P}_s]\,. \tag{16}$$

The right Blaschke factors $\mathbf{B}_s(p)$ do not coincide with the corresponding Blaschke factors $B_s(p)$ in the above product (14), due to the non-commutativity of the factors. We suggest here the procedure of construction of the factors $B_s(p)$ once the factors $\mathbf{B}_s(p)$ are given.

Assume that the factors $B_0, B_1, B_2, \ldots$ are ordered from the right to the left such that B_0 is the first factor from the right, B_1 is the second factor from the right, and so on. Denoting by $\mathbf{N}_s$, N_s the ranges of $\mathbf{P}_s$, P_s, respectively, and by v_s any vectors from $\mathbf{N}_s$, we can write down the following chain of equations

$$\begin{aligned}
&B_0(p_0) = \mathbf{B}_0(p_0), && N_0 = \mathbf{N}_0, && P_0 = \mathbf{P}_0,\\
&B_1(p_1)B_0(p_1)v_1 = 0, && \text{or} && N_1 = B_0(p_1)\mathbf{N}_1,\\
&B_2(p_2)B_1(p_2)B_0(p_2)v_2 = 0, && \text{or} && N_2 = B_1(p_2)B_0(p_2)\mathbf{N}_2,\\
&\qquad\ldots\\
&B_l(p_l)\cdots B_2(p_l)B_1(p_l)B_0(p_l)v_l = 0 && \text{or} && N_l = B_{l-1}(p_l)\cdots B_1(p_l)B_0(p_l)\mathbf{N}_l.
\end{aligned} \tag{17}$$

We obtain the chain of one-dimensional subspaces if each product $B_{l-1}(p_l)\cdots B_1(p_l)B_0(p_l)$ of Blaschke factors does not degenerate on the corresponding subspace $\mathbf{N}_l$:

$$B_{l-1}(p_l)\cdots B_1(p_l)B_0(p_l)v_l \neq 0. \tag{18}$$

Theorem 3.3. *The condition (18) of transformation of the rational form of the scattering matrix (13) into the Blashke product (14) is fulfilled for small values of the perturbation parameter ε. Hence the stationary scattering matrix (13) can be represented in the form of the Blaschke–Potapov product.*

Proof. For small values of the perturbation parameter the imaginary parts $\operatorname{Im} p_s$ of resonances are small, hence each term in the previous chain of equations can be rewritten in the form

$$\begin{aligned}
N_1 &= \left[I - \mathrm{i}\frac{2\operatorname{Im} p_0}{p_1 - \bar{p}_0}P_0\right]\mathbf{N}_1,\\
N_2 &= \left[I - \mathrm{i}\frac{2\operatorname{Im} p_1}{p_2 - \bar{p}_1}P_1\right]\left[I - \mathrm{i}\frac{2\operatorname{Im} p_0}{p_2 - \bar{p}_0}P_0\right]\mathbf{N}_2\\
&\ldots\\
N_l &= \left[I - \mathrm{i}\frac{2\operatorname{Im} p_{l-1}}{p_l - \bar{p}_{l-1}}P_{l-1}\right]\left[I - \mathrm{i}\frac{2\operatorname{Im} p_{l-2}}{p_l - \bar{p}_{l-2}}P_{l-2}\right]\left[I - \mathrm{i}\frac{2\operatorname{Im} p_0}{p_l - \bar{p}_0}P_0\right]\mathbf{N}_l.
\end{aligned}$$

Then due to small $\operatorname{Im} p_0$ the operator $\left[I - \mathrm{i}\frac{2\operatorname{Im} p_0}{p_1 - \bar{p}_0}P_0\right]$ is invertible and hence N_1 has the same dimension as $\mathbf{N}_1$. The projection P_1 exists. Then above argument may be applied to the second equation, to find P_2, and so on until all projections P_l are defined. □

Note that the structure of each Blaschke factor B_s shows that it has a zero at $p_s(\varepsilon)$, a pole at $\bar{p}_s(\varepsilon)$ and both of them approach the eigenvalue of the inner Hamiltonian when $\varepsilon \to 0$. The Blaschke factor B_0 is not analytic with respect to (ε, p) near $(0, \alpha_0)$ due to the convergence of the zero and the pole of the factor to the same point α_0. The Blaschke factor $B_s s \neq 0$ is analytic with respect to (ε, p) for small values of ε, since $|p - \alpha_s| > \delta > 0$ for p close to α_0. The whole scattering matrix (16) is not analytic with respect to the perturbation parameter near each eigenvalues α_s of the 'inner Hamiltonian ' A due to the presence of the non-analytic

factor B_s. Nevertheless, one may modify the perturbation procedure *locally*, eliminating, for instance, the non-analytic factor $S_0^\varepsilon := B_0$ via the 'jump-start': by introducing the *intermediate operator* $\mathcal{P}_0^\beta$, which is selected such that $S_0^\varepsilon = B_0$ is the scattering matrix to the pair $\left(\mathcal{P}_0^\varepsilon, \mathcal{P}\right)$. Then the scattering matrix can be presented as a product of the non-analytic, but explicit factor S_0^ε and the complementary analytic factor which is the scattering matrix to the pair $\left(\mathcal{P}_\varepsilon, \mathcal{P}_0^\varepsilon\right)$. The construction of the intermediate operator is described below under the assumption that *we know the resonance $p_0(\varepsilon)$ and the corresponding resonance root-vector v_ε exactly*. If we know $p_0(\varepsilon)$ only asymptotically, as a finite power expansion in the perturbation parameter, the corresponding approximate formulae are still valid, but the analyticity is lost. Practical corollaries from this observation will be discussed elsewhere.

Note that the real and imaginary parts of the resonance can be expanded into convergent power series of the real perturbation parameter ε, because of the analyticity of $p_0(\varepsilon)$. Hence the real and imaginary parts of the resonance are also real analytic functions of the perturbation parameter ε.

Theorem 3.4. *For given Blaschke factor*

$$B_\varepsilon = \left\{ \left[\frac{p - p_0(\varepsilon)}{p - \bar{p}_0(\varepsilon)} \right] P_0 + P_0^\perp \right\},$$

with the one-dimensional projection $P_0 = v_\varepsilon\rangle\langle v_\varepsilon$ consider the one-dimensional operator A_0^ε with the eigenvalue $\alpha_0 = \operatorname{Re} p_s(\varepsilon)$. Select some normalized deficiency vector $\mathbf{e} \in K$ and the boundary matrix

$$B = \begin{pmatrix} \beta_{00} & \beta_{01} \\ \beta_{10} & 0 \end{pmatrix}$$

mapping **e** *into E as*

$$\beta_{01} = \sqrt{2\frac{\operatorname{Im} p_0(\varepsilon)}{1+\alpha_0^2}}\, v_0(\varepsilon)\rangle\langle \mathbf{e}, *\rangle.$$

Choose $\beta_{00} = -\beta_{01}\alpha_0\beta_{01}^+$. Then the model described by theorems 2.2, and 2.3. has the scattering matrix B_ε. The scattering matrix and the scattered waves of the model are found based on the ansatz (3),

$$\mathbf{\Psi} = \begin{pmatrix} \Psi(x) \\ \frac{\alpha_0+\mathrm{i}}{\alpha_0-p}\eta_+ e \end{pmatrix}$$

via eliminating η_+ from the boundary conditions with $\eta_- = -\frac{1+\alpha_0 p}{\alpha_0-p}\eta_+$.

Proof. The scattering matrix for the constructed operator is calculated as in (7):

$$S(p) = \frac{2\mathrm{i} - \frac{1+\alpha_0^2}{\alpha_0-p}\beta_{01}\rangle\langle\beta_{01}}{2\mathrm{i} + \frac{1+\alpha_0^2}{\alpha_0-p}\beta_{01}\rangle\langle\beta_{01}}.$$

Multiplying by $(p-\alpha_0)$ and dividing through 2i, we transform the latter expression into

$$\frac{p-\alpha_0 - \mathrm{i}\frac{(1+\alpha^2)}{2}|\beta_{01}|^2}{p-\alpha_0 - \mathrm{i}\frac{(1+\alpha^2)}{2}|\beta_{01}|^2} P_0 + P_0^\perp = P_0^\perp + \frac{p - \left(\alpha_0 + \mathrm{i}|\beta_{01}|^2\frac{1+\alpha_0^2}{2}\right)}{p - \left(\alpha_0 - \mathrm{i}|\beta_{01}|^2\frac{1+\alpha_0^2}{2}\right)} P_0,$$

where $P_0^\perp$ is the orthogonal projection onto the complement of the null-space of the Blaschke factor B_ε. Substituting here the data $\alpha_0(\varepsilon) = \operatorname{Re} p_0(\varepsilon)$, $|\beta_{01}|^2\frac{1+\alpha_0^2}{2} = \operatorname{Im} p_0(\varepsilon)$, we obtain the Blaschke factor B_ε. □

Remark. Eliminating the variables $\eta_\pm$ of the inner space we obtain in the outer space $L_2(R, E)$ the spectral problem with the boundary conditions at the origin containing the spectral parameter p:

$$\xi_- = -\frac{2\,\mathrm{Im}\, p_0(\varepsilon)}{\mathrm{Re}\, p_0(\varepsilon) - p} P_0 \xi_+. \tag{19}$$

In the scalar case the boundary condition is just scalar. The corresponding stationary scattering matrix coincides with the Blaschke factor B_ε.

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The authors recognize the importance of the brave idea of I Prigogine concerning the intermediate operators in analytic perturbations of operators with continuous spectrum, and the connection of them with resonances.

Appendix

Let E_0 be a proper subspace of the finite-dimensional Hilbert space E, $\mathbf{P}_0$ be an orthogonal projection onto E_0 and $\mathbf{P}_0^\perp = I - \mathbf{P}_0$ be the projection onto the orthogonal complement $E_0^\perp = E \ominus E_0$. We state that the analytic matrix function m defined on the domain D_m has a *simple isolated right vector zero* at the point $p_0 \in D_m$, if it may be represented in a neighbourhood $U_0 \subset D_\mu$ as a product

$$m(p) = \mu_0(p)\big[(p - p_0)\mathbf{P}_0 + b\mathbf{P}_0^\perp\big] \tag{A.1}$$

with some non-zero constant b, the right orthogonal projection $\mathbf{P}_0$ onto the corresponding right root-subspace E_0 and an invertible near p_0 analytic matrix function $\mu_0(p)$

$$\mu_0(p) = \mu_0(p_0) + \frac{p - p_0}{1!}\mu_0'(p_0) + \cdots, \qquad \mathrm{Ker}\, \mu_0(p_0) = 0.$$

Multiple zeros are defined similarly to (A.1), with several right factors containing possibly different projections. One can define in a similar way the left vector zero and the corresponding left projection based on the factorization

$$m(p) = \big[(p - p_0)\mathbf{P}_0^+ + b\mathbf{P}_0^\perp\big]\mu_0^l(p), \tag{A.2}$$

with invertible $\mu_0^l(p_0)$. For finite-dimensional square matrix functions the left and right vector zeros coincide and $\dim \mathbf{P}_0 = \dim \mathbf{P}_0^+$ due to the Fredholm theorem. The vectors $\mathbf{e}_0, \mathbf{e}_0^+ \in \mathbf{N}_0, \mathbf{N}_0^+$ from the corresponding null-subspaces are called, respectively, right and left *root-vectors*, $m(p_0)\mathbf{e}_0 = 0$, $m^+(p_0)\mathbf{e}_0^+ = 0$. For vectors $\mathbf{e}_\perp$ from the complementary subspace $\mathbf{e}_\perp \in E_{0\perp}$ we have $m(p_0)e_\perp \neq 0$. Similarly, the *simple isolated vector pole* is defined: we state that the function m has a simple isolated vector pole at the point p_0 if it is represented as

$$m(p) = \mu\left[\frac{\mathbf{P}_0}{p - p_0} + b\mathbf{P}_0^\perp\right] \tag{A.3}$$

with a non-zero constant b, an orthogonal projection $\mathbf{P}_0$ onto proper subspace $\mathbf{N}_0 \subset E$, the complementary projection $\mathbf{P}_0^\perp$ and an analytic invertible function μ in a neighbourhood U_0 of the point $p_0 \in D_m$. Similarly the left poles are defined, which coincide with right poles in the finite-dimensional case. Both isolated poles and zeros of analytic matrix functions are

called in [36] *characteristic values* of the argument μ. The logarithmic residue of the function μ at the simple isolated zero or pole is defined as an integral of the logarithmic derivative $m'(p)m^{-1}(p)$ on a simple smooth curve $\Gamma_0 \subset U_0$ in anti-clockwise ('positive') direction around the characteristic value m_0:

$$I_{m,p_0} = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} m'(p)m^{-1}(p)\,\mathrm{d}p.$$

In [36] the *period* of the logarithmic derivative $m'(p)m^{-1}(p)$ on the simple cycle $\Gamma_0 \subset U_0$ containing no other characteristic points (zeros, poles) inside

$$M_{m,p_0} = \frac{1}{2\pi \mathrm{i}} \mathrm{Trace} \oint_{\Gamma_0} m'(p)m^{-1}(p)\,\mathrm{d}p.$$

is called the 'multiplicity' of the characteristic value. Straightforward calculation of the above integrals gives the following result, in the case of only two factors present in A.1.

$$\begin{aligned} I(m,\,p_0) &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} m'(p)m^{-1}(p)\,\mathrm{d}p \\ &= \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma_0} \mu(p)\mathbf{P}_0 \left[(p_0 - p)\mathbf{P}_0 + b\mathbf{P}_0^{\perp}\right]^{-1} \mu^{-1}(p)\,\mathrm{d}p \\ &= \frac{1}{2\pi \mathrm{i}} \mu(p_0) \oint_{\Gamma_0} \mathbf{P}_0 \left[(p_0 - p)\mathbf{P}_0 + b\mathbf{P}_0^{\perp}\right]^{-1} \mathrm{d}p\, \mu^{-1}(p_0) = \mu(p_0)\mathbf{P}_0\mu^{-1}(p_0), \end{aligned}$$

and

$$M_{m,p_0} = \pm \dim \mathbf{P}_0,$$

where the sign $\pm$ is defined by the type of the characteristic value: plus for zero, minus for pole. For general formula see [36].

Theorem A.1. *If two finite square matrices m, m_0 depend analytically on the parameter p in the disc D radius δ centred at the point p_0, and m_0 has only one characteristic point p_0 at the centre of the disc with the multiplicity M_0, both functions have no characteristic values on the circle $\Sigma_0 = \{p : |p - p_0| = \delta\}$, and the inequality*

$$max_{p\in\Sigma_0} \left\| m_0^{-1}(p)\left[m(p) - m_0(p)\right]\right\| < 1$$

is fulfilled, then the total multiplicity M_1 of characteristic values of the function m inside the circle Σ_0 is equal to the multiplicity M_0 of the characteristic value of the function m_0.

We actually need in the above text, section 3, a partial statement concerning the case when $M_0 = 1$. A formulation and proof of a much more general statement concerning analytic functions with multiple poles and zeros can be found in [36].

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